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IV.

THE FACTORIZATION OF THE HYPERGEOMETRIC EQUATION

by ERWIN SCHRÖDINGER.
(From the Dublin Institute for Advanced Studies.)

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As a sequel to investigations in factorizing ordinary homogeneous linear differential equations of the second order,* I here indicate a quadruple of factorizations of the hypergeometric equation, the one that determines Gauss's function $F(\alpha, \beta, \gamma, x)$, of which most of the functions occurring in physics are either special or limiting cases. The equation reads

$$x(1-x)y^{''} + [\gamma - (\alpha + \beta + 1)x]y^{'} - \alpha\beta y = 0 \ . \tag{1}$$

In physical applications x is usually restricted to

$$0 \leq x \leq 1 \ . \tag{2}$$

If by

$$\cos\theta = 2x - 1 \tag{3}$$

you introduce the independent variable θ (which by (2) would be restricted to

$$\pi \geq \theta \geq 0) \ , \tag{4}$$

you get

$$\frac{d^2y}{d\theta^2} + \frac{a\cos\theta + b}{\sin\theta}\frac{dy}{d\theta} + cy \ , \tag{5}$$

with

$$a = \alpha + \beta \ , \qquad b = \alpha + \beta + 1 - 2\gamma \ , \qquad c = -\alpha\beta \ . \tag{6}$$

*Proc. R.I.A. **46** A (1940) 9; *ibid*, (1941) 183.

If now you introduce the new *dependent* variable

$$z = (\sin\theta)^{\frac{a}{2}} \left(\tan\frac{\theta}{2}\right)^{\frac{b}{2}} y \ . \tag{7}$$

you obtain

$$\frac{d^2z}{d\theta^2} + \left[c + \frac{a^2}{4} - \frac{2b(a-1)\cos\theta + a^2 + b^2 - 2a}{4\sin^2\theta}\right] z = 0 \ . \tag{8}$$

This is readily factorized thus

$$\left(\frac{d}{d\theta} + \frac{C}{\sin\theta} + D\cot\theta\right)\left(\frac{d}{d\theta} - \frac{C}{\sin\theta} - D\cot\theta\right) z + Bz = 0 \ . \tag{9}$$

Comparing the coefficients the following *four* alternatives are offered: -

$$\begin{array}{llll}
(1) & B = c \ , & C = \frac{b}{2} \ , & D = \frac{a}{2} \ . \\
(2) & B = c + a - 1 \ , & C = -\frac{b}{2} \ , & D = 1 - \frac{a}{2} \ . \\
(3) & B = c + \frac{a^2}{4} - \frac{(b+1)^2}{4} \ , & C = \frac{a-1}{2} \ , & D = \frac{b+1}{2} \ . \\
(4) & B = c + \frac{a^2}{4} - \frac{(b-1)^2}{4} \ , & C = -\frac{a-1}{2} \ , & D = -\frac{b+1}{2} \ .
\end{array} \tag{10}$$

It will be realized that it is the factorizations (3) and (4) which lend themselves to the recurrent process described earlier. For they are obtained from one another by reversing the order of the first order operators in (9) and changing the value of b by ± 2, whilst a and c are unchanged. (From (6) that means that γ alone is changed, α and β remaining constant.) If the particular problem is such as to warrant $B \geq 0$, the recurrent process, in one or the other direction, must lead to a function for which

$$\left(\frac{d}{d\theta} - \frac{C}{\sin\theta} - D\cot\theta\right) z = 0 \ ,$$

and

$$B = 0 \ .$$

From this key-function the other solutions are obtained by repeated application of the *other* operator.

The factorizations (9), (10) must not be regarded as *the* factorizations of Gauss's equation. They belong to the particular *density*

$$\sigma = (\sin\theta)^a \left(\tan\frac{\theta}{2}\right)^b \ ,$$

according to (7). There are bound to exist others belonging to other density functions. They are *not* obtained just by a change of the dependent variable

$$\hat{z} = f(\theta) z \ .$$

This is a trivial transformation, which does not yield anything new.